On the Singular Solutions of Nonlinear Singular Partial Differential Equations (Asymptotic Analysis and Microlocal Analysis of PDE)

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On the Singular Solutions of Nonlinear Singular Partial Differential Equations

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Abstract

Let us consider the following nonlinear singular partial differential equation: $(t\partial_t)^m u = F(t,x,\{(t\partial_t)^j \partial_x^\alpha u\}_{j+|\alpha|\le m, j<m})$ in the complex domain. Denote by $\mathcal{S}_+$ [*resp.* $\mathcal{S}_{log}$] the set of all the solutions $u(t,x)$ with asymptotics $u(t,x)=O(|t|^a)$ [*resp.* $u(t,x)=O(1/|\log t|^a)$] (as $t \longrightarrow 0$ uniformly in x) for some $a>0$. Clearly $\mathcal{S}_{log} \supset \mathcal{S}_+$. The paper gives a sufficient condition for $\mathcal{S}_{log}=\mathcal{S}_+$ to be valid.

The paper deals with nonlinear singular partial differential equations of the form

$$\text{(E)} \qquad (t\partial/\partial t)^m u = F\left(t,x,\{(t\partial/\partial t)^j(\partial/\partial x)^\alpha u\}_{j+|\alpha|\le m, j<m}\right)$$

in the complex domain. In Gérard-Tahara [1] the author has determined all the singular solutions $u(t,x)$ of (E) under the condition that $u(t,x)=O(|t|^a)$ (as $t \longrightarrow 0$ uniformly in x) for some $a>0$.

The present paper investigates singular solutions $u(t,x)$ of (E) under a weaker condition that $u(t,x)=O(1/|\log t|^a)$ (as $t \longrightarrow 0$ uniformly in x) for some $a>0$.

§1. Equations.

Notations: $t\in \boldsymbol{C}$, $x=(x_1,\ldots,x_n)\in \boldsymbol{C}^n$, $\boldsymbol{N}=\{0,1,2,\ldots\}$, and $\boldsymbol{N}^*=\{1,2,\ldots\}$. For $\alpha=(\alpha_1,\ldots,\alpha_n)\in \boldsymbol{N}^n$ we write $|\alpha|=\alpha_1+\cdots+\alpha_n$ and

$$\left(\frac{\partial}{\partial x}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1}\cdots\left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}.$$

Let $m\in \boldsymbol{N}^*$, $N=\#\{(j,\alpha)\in \boldsymbol{N}\times\boldsymbol{N}^n\,;\, j+|\alpha|\le m, j<m\}$, and write the variable Z as

$$Z=\{Z_{j,\alpha}\}_{\substack{j+|\alpha|\le m\\ j<m}} \in \boldsymbol{C}^N.$$

Let $F(t,x,Z)$ be a function in the variables (t,x,Z) defined in a neighborhood of the origin $(0,0,0) \in \boldsymbol{C}_t \times \boldsymbol{C}_x^n \times \boldsymbol{C}_Z^N$, and assume the following:

(A$_1$) $F(t,x,Z)$ is holomorphic near $(0,0,0)$;

(A$_2$) $F(0,x,0) \equiv 0$ near $x=0$;

(A$_3$) $\dfrac{\partial F}{\partial Z_{j,\alpha}}(0,x,0) \equiv 0$ near $x=0$, if $|\alpha|>0$.

In this paper we always assume the conditions (A$_1$), (A$_2$), (A$_3$), and we will consider the following nonlinear partial differential equation

$$\text{(E)} \qquad \Big(t\frac{\partial}{\partial t}\Big)^m u = F\Big(t,x,\Big\{\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^\alpha u\Big\}_{\substack{j+|\alpha|\le m\\ j<m}}\Big)$$

with $u=u(t,x)$ as an unknown function.

For (E) we set

$$C(\lambda,x)=\lambda^m-\sum_{j<m}\frac{\partial F}{\partial Z_{j,0}}(0,x,0)\lambda^j$$

and denote by $\lambda_1(x),\ldots,\lambda_m(x)$ the roots of the equation $C(\lambda,x)=0$ in λ. These $\lambda_1(x),\ldots,\lambda_m(x)$ are called the *characteristic exponents* of (E).

The following is our basic problem:

Problem. Determine all kinds of local singularities which appear in the solutions of (E).

§2. Gérard-Tahara (1993)

Let us recall the result in Gérard-Tahara [1]. Denote:

- $\mathcal{R}(\boldsymbol{C}\setminus\{0\})$ denotes the universal covering space of $\boldsymbol{C}\setminus\{0\}$;
- $S_\theta=\{t\in\mathcal{R}(\boldsymbol{C}\setminus\{0\})\,;\,|\arg t|<\theta\}$;
- $S(\varepsilon(s))=\{t\in\mathcal{R}(\boldsymbol{C}\setminus\{0\})\,;\,0<|t|<\varepsilon(\arg t)\}$, where $\varepsilon(s)$ is a positive-valued continuous function on $\boldsymbol{R}_s$;
- $D_r=\{x\in\boldsymbol{C}^n\,;\,|x|\le r\}$;
- $\boldsymbol{C}\{x\}$ denotes the ring of convergent power series in x, or equivalently the ring of germs of holomorphic functions at the origin of $\boldsymbol{C}^n$.

Definition 1. We denote by $\widetilde{\mathcal{O}}_+$ the set of all $u(t,x)$ satisfying the following conditions i) and ii):

i) $u(t,x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_r$ for some positive-valued continuous function $\varepsilon(s)$ and some $r > 0$;

ii) there is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{|x|\le r} \left|u(t,x)\right| = O\Big(|t|^a\Big) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

For the characteristic exponents $\lambda_1(x), \dots, \lambda_m(x)$, we set

$$\mu = \#\{\, i \,; \mathrm{Re}\lambda_i(0) > 0\}.$$

When $\mu = 0$, this is equivalent to the fact that $\mathrm{Re}\lambda_i(0) \le 0$ for all $i = 1, \dots, m$. When $\mu \ge 1$, by a renumeration we may assume

$$\tag{1.1} \begin{cases} \mathrm{Re}\lambda_i(0) > 0 & \text{for } 1 \le i \le \mu, \\ \mathrm{Re}\lambda_i(0) \le 0 & \text{for } \mu + 1 \le i \le m. \end{cases}$$

Then we already have:

Theorem 1 (Gérard-Tahara [1]). *Denote by $\mathcal{S}_+$ the set of all $\widetilde{\mathcal{O}}_+$-solutions of* (E). *Then we have:*

(I) *When $\mu = 0$, we have $\mathcal{S}_+ = \{u_0\}$ where $u_0 = u_0(t,x)$ is the unique holomorphic solution of* (E) *satisfying $u_0(0,x) \equiv 0$.*

(II) *When $\mu \ge 1$, under* (1.1) *and the following additional conditions*

1) $\lambda_i(0) \ne \lambda_j(0)$ *for* $1 \le i \ne j \le \mu$,
2) $C(1,0) \ne 0$,
3) $C(i + j_1\lambda_1(0) + \cdots + j_\mu\lambda_\mu(0), 0) \ne 0$ *for any* $(i,j) \in \boldsymbol{N} \times \boldsymbol{N}^\mu$ *satisfying* $i + |j| \ge 2$ (*where* $j = (j_1, \dots, j_\mu)$),

we have

$$\mathcal{S}_+ = \Big\{U(\phi_1, \dots, \phi_\mu)\,;\, (\phi_1, \dots, \phi_\mu) \in (\boldsymbol{C}\{x\})^\mu\Big\},$$

where $U(\phi_1, \dots, \phi_\mu)$ is an $\widetilde{\mathcal{O}}_+$-solution of (E) determined by $(\phi_1, \dots, \phi_\mu) \in (\boldsymbol{C}\{x\})^\mu$ and having the expansion of the following form:

$$\begin{aligned} U(\phi_1, \dots, \phi_\mu) \;&=\; \sum_{i \ge 1} u_i(x)\, t^i \\ &\quad + \phi_1(x)\, t^{\lambda_1(x)} + \cdots + \phi_\mu(x)\, t^{\lambda_\mu(x)} \\ &\quad + \sum_{\substack{i+2m|j| \ge k+2m \\ |j| \ge 1 \\ (i,|j|) \ne (0,1)}} \varphi_{i,j,k}(x)\, t^{i + j_1\lambda_1(x) + \cdots + j_\mu\lambda_\mu(x)} (\log t)^k. \end{aligned}$$

§3. Problems.

In Theorem 1 we have restricted ourselves to the study of singular solutions in $\tilde{\mathcal{O}}_+$. But, there seems to be a possibility that (E) has singular solutions which do not belong in the class $\tilde{\mathcal{O}}_+$, as is seen in the following example.

Example 1. The equation

$$t\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right)^k$$

(where $(t,x) \in \boldsymbol{C}^2$ and $k \in \boldsymbol{N}^*$) has a family of singular solutions

$$u(t,x) = \left(\frac{1}{k}\right)^{1/k} \frac{x+\alpha}{(c-\log t)^{1/k}}, \qquad \alpha, c \in \boldsymbol{C},$$

which do not belong in the class $\tilde{\mathcal{O}}_+$.

In order to include this kind of singular solutions in our framework, we introduce the following new class of singular solutions:

Definition 2. We denote by $\tilde{\mathcal{O}}_{log}$ the set of all $u(t,x)$ satisfying the following conditions i) and ii):
i) $u(t,x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_r$ for some positive-valued continuous function $\varepsilon(s)$ and some $r > 0$;
ii) there is an $a > 0$ such that for any $\theta > 0$ we have

$$\max_{|x|\le r} \left|u(t,x)\right| = O\left(\frac{1}{|\log t|^a}\right) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

Clearly we have $\tilde{\mathcal{O}}_{log} \supset \tilde{\mathcal{O}}_+$. Therefore, if we denote by $\mathcal{S}_{log}$ the set of all $\tilde{\mathcal{O}}_{log}$-solutions of (E), we have $\mathcal{S}_{log} \supset \mathcal{S}_+$.

We will say that $u(t,x)$ is a solution with temperate singularities if $u(t,x) \in \mathcal{S}_+$, and that $u(t,x)$ is a solution with logarithmic singularities if $u(t,x) \in \mathcal{S}_{log} \setminus \mathcal{S}_+$.

Our next problems can be set up as follows:

Problem 1. When does $\mathcal{S}_{log} = \mathcal{S}_+$ hold ?
Problem 2. When does $\mathcal{S}_{log} \neq \mathcal{S}_+$ hold ?

Note that the problem 1 asserts that new singular solutions do not appear and that the problem 2 asserts that new singular solutions really appear in the solutions of (E).

In this paper we will give a partial answer and a conjecture on the problem 1. The problem 2 will be discussed in the forthcoming paper.

§4. A result and a conjecture.

In this section we will give a result on the problem 1 in a general form.

A function $\mu(t)$ on $(0,T)$ is called *a weight function* if it satisfies the following conditions $\mu_1) \sim \mu_3)$:

$$\begin{aligned} &\mu_1) \quad \mu(t) \in C^0((0,T)), \\ &\mu_2) \quad \mu(t) > 0 \text{ on } (0,T) \text{ and } \mu(t) \text{ is increasing in } t, \\ &\mu_3) \quad \int_0^T \frac{\mu(s)}{s} ds < \infty. \end{aligned}$$

By $\mu_2)$ and $\mu_3)$ the condition $\mu(t) \longrightarrow 0$ (as $t \longrightarrow +0$) is clear. In this paper we impose the additional condition on $\mu(t)$:

$$\mu(t) \in C^1((0,T)) \quad \text{and} \quad \Big(t\frac{d\mu}{dt}\Big)(t) = o(\mu(t)) \quad (\text{as } t \longrightarrow +0). \tag{4.1}$$

The following functions are typical examples:

$$\mu(t) = \frac{1}{(-\log t)^b}, \quad \frac{1}{(-\log t)(\log(-\log t))^c}$$

with $b > 1$, $c > 1$. Note that the function $\mu(t) = t^d$ with $d > 0$ does not satisfy the condition (4.1).

Definition 3. Let $\mu(t)$ be a weight function.

(1) For $a > 0$ we denote by $\tilde{\mathcal{O}}_a(\mu(t))$ the set of all $u(t,x)$ satisfying the following conditions i) and ii):

i) $u(t,x)$ is a holomorphic function on $S(\varepsilon(s)) \times D_r$ for some positive-valued continuous function $\varepsilon(s)$ and some $r > 0$;

ii) for any $\theta > 0$ we have

$$\max_{|x| \le r} \big|u(t,x)\big| = O\Big(\mu(|t|)^a\Big) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta).$$

(2) We define $\tilde{\mathcal{O}}_+(\mu(t))$ by

$$\tilde{\mathcal{O}}_+(\mu(t)) = \bigcup_{a>0} \tilde{\mathcal{O}}_a(\mu(t)).$$

Lemma 1. (1) $\tilde{\mathcal{O}}_{log} = \tilde{\mathcal{O}}_+(\mu(t))$ *if* $\mu(t) = 1/(-\log t)^b$ *with* $b > 1$.
(2) *If* $\mu(t)$ *satisfies* (4.1) *we have* $\tilde{\mathcal{O}}_+ \subset \tilde{\mathcal{O}}_1(\mu(t))$ $(\subset \tilde{\mathcal{O}}_+(\mu(t)))$.

Proof. (1) is clear. (2) is verified as follows. By (4.1), for any $\varepsilon > 0$ there is a $\delta > 0$ such that $t\mu_t'(t) \le \varepsilon\mu(t)$ holds on $(0,\delta]$ and therefore we have

$$\frac{d}{dt}(t^{-\varepsilon}\mu(t)) \le 0 \quad \text{for} \quad 0 < t \le \delta.$$

Integrating this from t to δ we have

$$\delta^{-\varepsilon}\mu(\delta) \le t^{-\varepsilon}\mu(t) \quad \text{for} \quad 0 < t \le \delta$$

and so

$$\left(\frac{\mu(\delta)}{\delta^{\varepsilon}}\right) t^{\varepsilon} \le \mu(t) \quad \text{for} \quad 0 < t \le \delta. \tag{4.2}$$

Since $\varepsilon > 0$ is arbitrary, (4.2) leads us to the conclusion of (2). □

Denote by $\mathcal{S}_+(\mu(t))$ (resp. $\mathcal{S}_a(\mu(t))$) the set of all $\widetilde{\mathcal{O}}_+(\mu(t))$-solutions of (E) (resp. $\widetilde{\mathcal{O}}_a(\mu(t))$-solutions of (E)). By (2) of Lemma 1 we have

$$\mathcal{S}_+ \subset \mathcal{S}_1(\mu(t)) \subset \mathcal{S}_+(\mu(t)).$$

The following theorem gives a sufficient condition for $\mathcal{S}_+(\mu(t)) = \mathcal{S}_+$ to be valid.

Theorem 2. *Let $\mu(t)$ be a weight function satisfying* (4.1). *Then, $\mathcal{S}_+(\mu(t)) = \mathcal{S}_+$ is valid if*

$$\operatorname{Re}\lambda_i(0) < 0 \quad \textit{for all} \quad i = 1, \ldots, m \tag{4.3}$$

or if

$$\operatorname{Re}\lambda_i(0) > 0 \quad \textit{for all} \quad i = 1, \ldots, m. \tag{4.4}$$

In the case (4.3), by Theorem 1 we have $\mathcal{S}_+ = \{u_0\}$ and therefore the condition $\mathcal{S}_+(\mu(t)) = \mathcal{S}_+$ is equivalent to the fact that the local uniqueness of the solution is valid in $\mathcal{S}_+(\mu(t))$ which is already proved in Tahara [4],[5].

In the case (4.4) the proof of Theorem 2 consists of the following two parts:

C_1) if $u \in \mathcal{S}_+(\mu(t))$ we have $u \in \mathcal{S}_m(\mu(t))$;

C_2) if $u \in \mathcal{S}_m(\mu(t))$ we have $u \in \mathcal{S}_+$.

The proofs of thses C_1) and C_2) will be published in Tahara [6].

Corollary. *If* (4.3) *or* (4.4) *holds, we have $\mathcal{S}_{log} = \mathcal{S}_+$.*

Remark. The author believes that the following conjecture is true, though at present he has no idea to prove this conjecture:

Conjecture. *$\mathcal{S}_{log} = \mathcal{S}_+$ is valid if*

$$\operatorname{Re}\lambda_i(0) \neq 0 \quad \textit{for all} \quad i = 1, \ldots, m. \tag{4.5}$$

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